

# $B_h[g]$ SEQUENCES

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ABSTRACT. We give new upper and lower bounds for  $F_h(g, N)$ , the maximum size of a  $B_h[g]$  sequence contained in  $[1, N]$ . We prove

$$F_h(g, N) \leq (\sqrt{3}hh!gN)^{1/h},$$

and for any  $\epsilon > 0$  and  $g > g(\epsilon, h)$ ,

$$F_h(g, N) \geq \left( (1 - \epsilon) \sqrt{\frac{\pi}{6}} \sqrt{h}gN \right)^{1/h} + o(N^{1/h}).$$

## 1. INTRODUCTION

Given a sequence of integers  $A$ , we define  $R_h(A; k)$  as the number of representations of  $k$  as the sum of  $h$  elements of  $A$ ,

$$R_h(A; k) = \#\{k = a_1 + \cdots + a_h; a_1 \leq \cdots \leq a_h, a_i \in A\},$$

and we say that a sequence of integers  $A$  is a  $B_h[g]$  sequence if  $R_h(A; k) \leq g$  for any integer  $k$ . Sidon was led to consider such sequences in connection with the theory of Fourier series.  $B_2[1]$  sequences are also called Sidon sequences.

It is a major problem giving good estimates for  $F_h(g, N)$ , the maximum size of  $B_h[g]$  sequences contained in  $\{1, \dots, N\}$ . See [H-R] for a classical reference about this topic, and [S-S] and [K] for recent surveys.

By a trivial counting argument we obtain the upper bound

$$F_h(g, N) \leq (hh!gN)^{1/h}.$$

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For  $g = 1$ , Erdős and Turan [E-T] proved  $F_2(1, N) \leq N^{1/2} + O(N^{1/4})$ . See [C1] for new upper bounds for  $F_h(1, N)$ ,  $h \geq 3$ .

On the other hand, Erdős observed in an addendum to [E-T] that a construction of J. Singer in [S] gives  $F_2(1, N) \geq N^{1/2} + o(N^{1/2})$ . (See [R] for an easy construction). Later R. C. Bose and S. Chowla [B-Ch] were able to prove  $F_h(1, N) \geq N^{1/h} + o(N^{1/h})$  for any integer  $h$ .

When  $g > 1$  is more difficult to obtain good estimates for  $F_h(g, N)$ . The first author [C2] and M. Helm [H], independently proved  $F_2(2, N) \leq \sqrt{6N} + 1$ . In [C-R-T] nontrivial upper bounds were proved for  $F_h(g, N)$ :

$$F_2(g, N) \leq \left( \frac{4}{1 + \frac{1}{(\pi/2+1)^2}} gN \right)^{1/2} + o(N^{1/2}),$$

$$F_h(g, N) \leq \frac{1}{(1 + \cos^h(\pi/h))^{1/h}} (hh!gN)^{1/h} + o(N^{1/h}), \quad h \geq 3.$$

Also N. Alon (see [K]) has obtained non trivial upper bounds for large  $h$  by exploiting the “concentration” of the sums  $a_1 + \dots + a_h$  around their mean, using the Chebyshev inequality. He gets

$$F_h(g, N) \leq \left( 3^{3/2} \sqrt{h} h! gN \right)^{1/h} + o(N^{1/h}).$$

In section 2 we get new upper bounds which improve the previous ones for  $h \geq 7$ . In particular we prove

**Theorem 1.1.**

$$F_h(g, N) \leq (\sqrt{3} h h! gN)^{1/h}.$$

In relation to the lower bounds, a construction of M. Kolountzakis [K] of  $B_2[2]$  sequences gives  $F_2(2, N) \geq \sqrt{2N} + o(N^{1/2})$ . In [C-R-T] it is proved that  $F_2(g, N) \geq \sqrt{\frac{g+[g/2]}{g+2[g/2]}} N^{1/2} + o(N^{1/2})$ .

Recently Lindstrom [L] generalized the argument of [K] to prove  $F_h(g, N) \geq (gN)^{1/h} + o(N^{1/h})$  for  $g = m^{h-1}$ ,  $m \geq 2$ .

In section 3 we obtain new lower bounds, improving the previous ones for  $h \geq 3$ .

**Theorem 1.2.** *Let  $h$  be an integer fixed. For any  $\epsilon > 0$ , and for any  $g > g(\epsilon, h)$*

$$F_h(g, N) \geq \left( (1 - \epsilon) \sqrt{\frac{\pi}{6}} \sqrt{h} g N \right)^{1/h} + o(N^{1/h}),$$

when  $N \rightarrow \infty$ .

In our construction it is needed  $g$  to be big enough. This situation already appears in [L, (1.6)] and in fact our construction recover Lindstrom result.

## 2. UPPER BOUNDS

In this section we note that the use of Chebyshev inequality is wasteful in Alon's argument. Instead of it we obtain a lower bound for the variance by observing that it cannot be smaller than the case where the integers are as compressed as possible.

*Proof of Theorem 1.1.*

Suppose that  $A \subset [1, N]$  is a  $B_h[g]$  sequence. Let the random variable  $Y$  be defined by  $Y = X_1 + \dots + X_h$ , where the  $X_j$  are independent random variables uniformly distributed in  $A$ . We can obtain an upper bound in an easy way:

$$E((Y - \bar{Y})^2) = hE((X - \bar{X})^2) \leq hE((X - (N+1)/2)^2) \leq h \frac{(N-1)^2}{4}.$$

In order to estimate  $E((Y - \bar{Y})^2)$  we consider the multiset  $hA = \{a_1 + \dots + a_h; a_i \in A\} = \{s_i : i = 1, \dots, k\}$ , where  $k = |A|^h$ .

$$|A|^h E((Y - \bar{Y})^2) = \sum_{s_i \in hA} (s_i - \bar{Y})^2.$$

The minimum value of the variance of a set happens when the elements are as close as possible. We observe that the  $s_i$ 's take integer values which appear, at most,  $gh!$  times (because  $A$  is a  $B_h[g]$  sequence of integers). Then, the variance of  $hA$  is not less than the variance of the multiset

$L = \{\overbrace{1, \dots, 1}^{h!g \text{ times}}, 2, \dots, 2, \dots, l, \dots, l\}$ , where  $l = \lfloor k/gh! \rfloor$ . Hence,

$$|A|^h E((Y - \bar{Y})^2) \geq \sum_{x \in L} (x - \bar{x})^2 = \sum_{x \in L} x^2 - |L| \bar{x}^2$$

$$\begin{aligned}
&= gh! \sum_{k=1}^l k^2 - gh!l \left( \frac{l+1}{2} \right)^2 = gh! \frac{l(l+1)(2l+1)}{6} - gh! \frac{l(l+1)^2}{4} \\
&= gh! \frac{l(l+1)(l+2)}{12} \geq \frac{k^3}{12(gh!)^2} - \frac{k}{12} \geq |A|^h \left( \frac{|A|^{2h}}{12(gh!)^2} - \frac{1}{12} \right).
\end{aligned}$$

Then we have proved

$$\frac{1}{12} \left( \frac{|A|^{2h}}{(gh!)^2} - 1 \right) \leq h \frac{(N-1)^2}{4},$$

which implies

$$|A| \leq ((gh!)^2(3h(N-1)^2 + 1))^{1/2h} \leq (\sqrt{3}hh!gN)^{1/h}.$$

□

### 3. LOWER BOUNDS

Now we are interested in  $B_h[g]$  sequences as dense as possible. We will establish a generalization of Theorem 2.1 of [C-R-T] for any integer  $h$ .

The proof will go as Theorem 2.1 in [C-R-T]. So, first of all we will need the analogous definitions as 2.1 and 2.2 in [C-R-T] for this general context.

**Definition 3.1.** *We say that  $A$  satisfies the  $B_h^*[g]$  condition if the equation  $a_1 + \dots + a_h = k$  has at most  $g$  solutions for any  $k$ , counting different those in distinct order.*

**Definition 3.2.** *We say that a sequence of integers  $C = \{c_i\}$  is a  $B_h \pmod{m}$  sequence if  $c_{i_1} + \dots + c_{i_h} = c_{j_1} + \dots + c_{j_h} \pmod{m}$  implies  $\{c_{i_1}, \dots, c_{i_h}\} = \{c_{j_1}, \dots, c_{j_h}\}$ .*

Now we can establish the lemma generalizing Lemma 2.2 in [C-R-T].

**Lemma 3.1.** *If  $A = \{a_i\}$  satisfies the  $B_h^*[g]$  condition, and  $C$  is a  $B_h \pmod{m}$  sequence, then  $B = \cup_{i=0}^k (C + ma_i)$  is a  $B_h[g]$  sequence.*

*Proof.* Suppose  $b_{1,1} + \dots + b_{h,1} = \dots = b_{1,g+1} + \dots + b_{h,g+1}$  for  $b_{i,j} \in B$ . We can write  $b_{i,j} = c_{i,j} + ma_{i,j}$  for some  $c_{i,j} \in C$  and  $a_{i,j} \in A$ . Let us order  $b_{i,j}$  such that, for any  $i$  and  $j$ ,  $c_{i,j} \leq c_{i+1,j}$ . Then, since

$c_{1,1} + \dots + c_{h,1} \equiv c_{1,j} + \dots + c_{h,j} \pmod{m}$  for any  $1 \leq j \leq g+1$  we have that all the sets  $\{c_{1,j}, \dots, c_{h,j}\}$  are the same. Moreover the elements are ordered and so  $c_{i,j} = c_{i,1}$  for every  $i, j$ . This implies further that all the  $g+1$  sums  $a_{1,j} + \dots + a_{h,j}$  are equal hence, for some  $j, j'$  we have  $a_{i,j} = a_{i,j'}$  for any  $1 \leq i \leq h$ . Both together give us, for these  $j, j'$ , that  $b_{i,j} = b_{i,j'}$  for any  $1 \leq i \leq h$ .  $\square$

In order to use Lemma 3.1, we have to find convenient sequences  $C$  and  $A$  on those conditions.

It is known [p. 81, H-R], that for  $m = p^h - 1$ ,  $p$  prime, there exists a  $B_h \pmod{m}$  sequence  $C_m \subset [1, m]$  with cardinal  $|C_m| = p$ .

On the other hand we can choose the trivial  $A_n = \{0, 1, \dots, n-1\}$  to get our bounds. Our next step is to find the greatest  $n$  so that  $A_n$  satisfies the  $B_h^*[g]$  condition. Let us call  $n(g, h)$  to this  $n$ . We have

**Proposition 3.1.**  $F_h(g, N) \geq n(g, h)^{1-1/h} N^{1/h} + o(N^{1/h})$ .

*Proof.* Let us take a prime  $p$  such that  $n(g, h)(p^h - 1) = N + o(N)$ . Now we apply Lemma 3.1 with  $m = p^h - 1$ ,  $A = A_{n(g, h)}$ . Then  $B \subset [1, n(g, h)m]$  and  $|B| = n(g, h)p$ .  $\square$

In order to estimate  $n(g, h)$  we define

$$r_h(n, k) = \#\{k = a_1 + \dots + a_h : 0 \leq a_i \leq n-1\}$$

and  $M_h(n) = \max_k r_h(n, k)$ . Then  $n(g, h)$  is the greatest  $n$  such that  $M_h(n) \leq g$ .

**Proposition 3.2.**  $M_h(n) \sim n^{h-1} \frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^h dt$

*Proof.* The obvious inequality  $M_h(2m-1) \leq M_h(2m) \leq M_h(2m+1)$  allows us to reduce the proof to  $n = 2m+1$ , odd. In this case we can write  $r_h(n, k) = \#\{k - hm = a_1 + \dots + a_h : -m \leq a_i \leq m\}$ . It is now trivial to deduce

$$r_h(n, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{j=-m}^m e^{ij\theta} \right)^h e^{-i(k-hm)\theta} d\theta,$$

by expanding the  $h$ -power of the Dirichlet kernel. Hence, by a simple change of variables

$$r_h(n, k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{\sin((m+1/2)\theta)}{\sin(\theta/2)} \right)^h e^{-i(k-hm)\theta} d\theta$$

$$\begin{aligned}
&= \frac{1}{\pi} (2m+1)^{h-1} \int_{-\pi(m+1/2)}^{\pi(m+1/2)} \left( \frac{\sin t}{t} \right)^h \left( \frac{t/(2m+1)}{\sin(t/(2m+1))} \right)^h e^{-i \frac{k-hm}{m+1/2} t} dt \\
&= \frac{1}{\pi} (2m+1)^{h-1} \int_{-\infty}^{\infty} \left( \frac{\sin t}{t} \right)^h e^{-i \frac{k-hm}{m+1/2} t} dt + o(m^{h-1}).
\end{aligned}$$

Now observe that  $\frac{\sin t}{t}$  is the fourier transform of the characteristic function of the interval  $[-1, 1]$ . Then  $\int_{-\infty}^{\infty} \left( \frac{\sin t}{t} \right)^h e^{-ixt} dt$  is the value of the  $h$ -convolution of  $\chi_{[-1,1]}$  at  $x$ , which is maximum at  $x = 0$  and so  $M_h(n) = \max_k r_h(n, k) \sim n^{h-1} \frac{2}{\pi} \int_0^{\infty} \left( \frac{\sin t}{t} \right)^h dt$ .  $\square$

**Proposition 3.3.** *Let  $h$  be an integer fixed. For every  $\epsilon > 0$  and for every  $g > g(\epsilon, h)$  we have*

$$F_h(g, N) \geq \left( (1 - \epsilon) \frac{g}{m_h} N \right)^{1/h} + o(N^{1/h}).$$

where  $m_h = \frac{2}{\pi} \int_0^{\infty} \left( \frac{\sin t}{t} \right)^h dt$ .

*Proof.* It is consequence of Proposition 3.1 and 3.2  $\square$

*Proof of Theorem 1.2.* We only need to study the behaviour of  $m_h$ . The upper bound  $m_h \leq \sqrt{6/\pi h}$  for  $h \geq 100$  follows from the corresponding for  $J_p$  (1) in [3.3, L-N]. For  $3 \leq h \leq 100$  we get the bound by computing the explicit formula (17), (see (15)), of  $[N]$

$$m_h = \frac{1}{(h-1)!} \sum_{j < h/2} (h/2 - j)^{h-1} \binom{h}{j} (-1)^j.$$

In particular we get for the first few values

$$\begin{aligned}
m_3 &= \frac{3}{4}, & m_4 &= \frac{2}{3}, & m_5 &= \frac{115}{192}, & m_6 &= \frac{11}{20}, & m_7 &= \frac{5587}{11520} \\
m_8 &= \frac{151}{315}, & m_9 &= \frac{259723}{573550}, & m_{10} &= \frac{15619}{36288}.
\end{aligned}$$

The case  $h = 2$  is covered in [C-R-T].  $\square$

**Remark 1.** It is possible to find an explicit formula for  $r_h(n, k)$  by using its generating function  $\left( \sum_{k=0}^{n-1} x^k \right)^h = \sum_k r_h(n, k) x^k$  together

with  $\sum x^n = \frac{1}{1-x}$ . Moreover, one can prove that its maximum  $M_h(n)$  is attained at the mean,  $k_h = (n-1)h/2$  or  $k_h = ((n-1)h+1)/2$ . In this way one can obtain the explicit expression

$$M_h(n) = \sum_{j=0}^h \binom{\frac{n-1}{2}h + \delta - nj + h - 1}{h-1} \binom{h}{j} (-1)^j,$$

where  $\delta = 0$  or  $1/2$  depending on the parity of  $(n-1)h$ , which is useful for small values of  $h$ .

For example, for  $h = 3$  we obtain  $M_3(n) = \left\lfloor \frac{3n^2+1}{4} \right\rfloor$ , and Proposition 3.1 gives the more precise estimate

$$F_3(g, N) \geq \left( \left[ \sqrt{\frac{4g}{3}} \right]^2 N \right)^{1/3} + o(N^{1/3}).$$

**Remark 2.** Induction in  $r_h(n, k) = \sum_{j=0}^{n-1} r_{h-1}(n, k-j)$  immediately implies  $M_h(n) \leq n^{h-1}$ . This and Proposition 3.1 recover Lindstrom result.

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